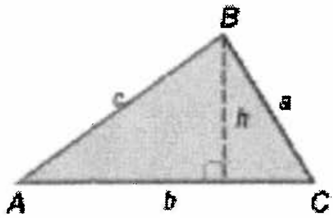
The Law of Sines

Name ______________________
Date ______________ Block _____

Area of a Triangle

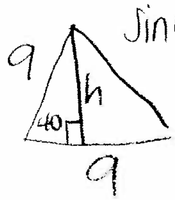


For the given triangle, $\sin A = \frac{h}{c}$.

Therefore, $h = c \sin A$.

Area of a triangle $= \frac{1}{2} \bullet \text{base} \bullet \text{height}$

$A = \frac{1}{2} \bullet b \bullet c \sin A$



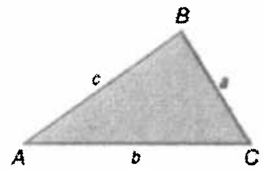
$\sin 40 = \frac{h}{9}$

SAS!

$\frac{1}{2}(9)9\sin 40)$
$\frac{1}{2}(b) \quad h$

Key Concept **Area of a Triangle**

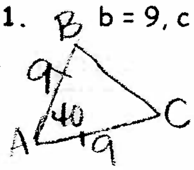
Words The area of a triangle is one half the product of the lengths of two sides and the sine of their included angle.



Symbols Area $= \frac{1}{2}bc \sin A$ Area $= \frac{1}{2}ac \sin B$ Area $= \frac{1}{2}ab \sin C$

Find the area of each triangle.

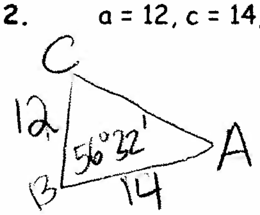
1. b = 9, c = 9, A = 40°



$\frac{1}{2} 9 \cdot 9 \cdot \sin 40$

26.03

2. a = 12, c = 14, B = 56°32′



$\frac{1}{2}(12)(14)(\sin 56°32')$

70.07

Law of Sines

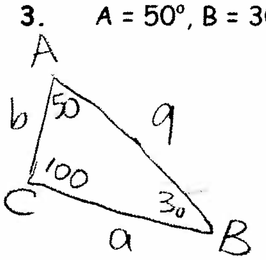
LAW of SINES: $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$

- **The Law of Sines is derived from the area formula.**
- **This law is used to solve triangles, including non-right triangles.**
- **The Law of Sines can be used when you know either one of the following:**
  - the measures of 2 angles and the lengths of one of the sides (AAS or ASA cases)
  - the measures of 2 sides and the angle opposite one of them (SSA case) – *see page 2 for details*

Solve each triangle described.

AAS

3. A = 50°, B = 30°, c = 9



$\frac{\sin 50}{a} = \frac{\sin 30}{b} = \frac{\sin 100}{9}$

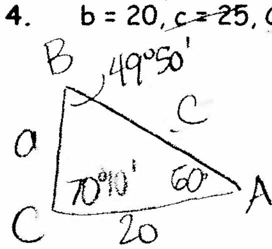
$a = \frac{9\sin 50}{\sin 100} \approx 7.00$

$b = \frac{9\sin 30}{\sin 100} \approx 4.57$

ASA

A = 60°

4. b = 20, ~~c = 25~~, C = 70°10′



$\frac{\sin 60}{a} = \frac{\sin 49°50'}{20} = \frac{\sin 70°10'}{c}$

$a = \frac{20\sin 60}{\sin 49°50'} \approx 22.67$

$c = \frac{20\sin 70°10'}{\sin 49°50'} \approx 24.62$

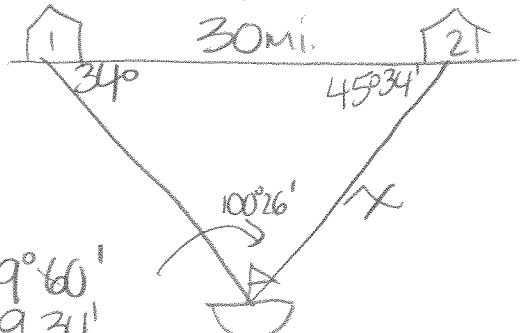
B = 179°60′
− 130°10′
49°50′

Applications

Solve each problem using the Law of Sines.

5. A ship is sighted at sea from two observation points on the coastline, which are 30 miles apart. The angle between the coastline and the line between the ship and the first observation point measures 34°. The angle between the coastline and the line between the ship and the second observation point measures 45° 34′. How far is the ship from the second observation point?

AAS

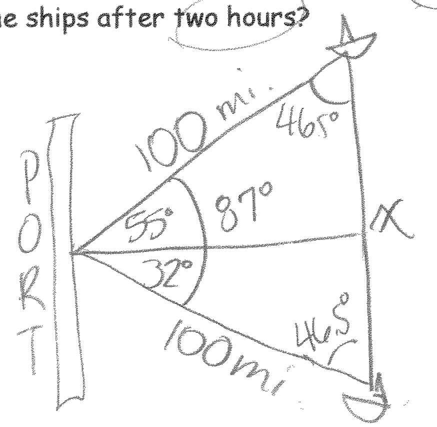


$$\begin{array}{r} 179°60' \\ -\ 79°34' \\ \hline 100°26' \end{array}$$

$$\frac{\sin 34}{x} = \frac{\sin 100°26'}{30}$$

$$x = \frac{30 \sin 34}{\sin 100°26'} \approx 17.06 \text{ miles}$$

6. Two ships leave port both traveling 50 mph. One travels 55° NE, and the other 32° SE. How far apart are the ships after two hours?

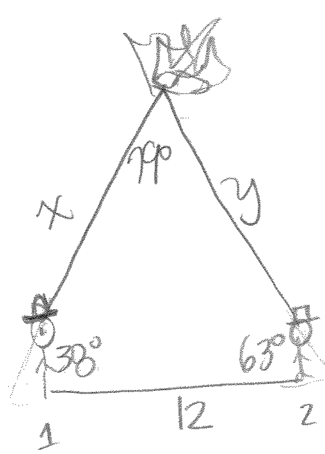


$$D = r \cdot t = 50 \cdot 2$$

$$\frac{\sin 87}{x} = \frac{\sin 46.5}{100}$$

$$x = \frac{100 \sin 87}{\sin 46.5} \approx 137.67 \text{ miles}$$

7. Two forest rangers, 12 miles from each other on a straight service road, both sight an illegal bonfire away from the road. After communicating through radios, they determine the fire is between them. The first line of sight to the fire makes an angle of 38° with the road and the second ranger's line of sight to the fire makes a 63° angle with the road. How far away is the fire from each ranger?



$$\frac{\sin 79}{12} = \frac{\sin 38}{y} = \frac{\sin 63}{x}$$

$$x = \frac{12 \sin 63}{\sin 79} \approx 10.89 \text{ miles}$$

$$y = \frac{12 \sin 38}{\sin 79} \approx 7.53 \text{ miles}$$

Law of Sines: The Ambiguous Case

http://www.regentsprep.org/Regents/math/algtrig/ATT12/lawofsinesAmbiguous.htm

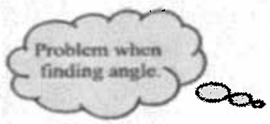
In Geometry, we found that we could prove two triangles congruent using:

SAS - Side, Angle, Side
ASA - Angle, Side, Angle
AAS - Angle, Angle, Side
SSS - Side, Side, Side
HL - Hypotenuse Leg for Right Triangles.

> We also discovered that SSA did not work to prove triangles congruent. We politely called it the Donkey Theorem ; -)

By definition, the word **ambiguous** means *open to two or more interpretations.* Such is the case for certain solutions when working with the Law of Sines.

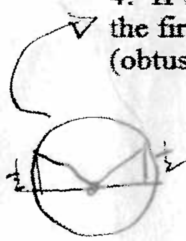
- If you are given two angles and one side (ASA or AAS), the Law of Sines will nicely provide you with ONE solution for a missing side.
- Unfortunately, the Law of Sines has a problem dealing with SSA. If you are given two sides and one angle (where you must *find an angle*), the Law of Sines could possibly provide you with one or more solutions, or even no solution.



Before we investigate this situation, there are a few facts we need to remember.

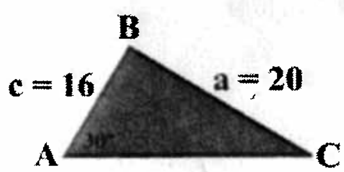
Facts we need to remember:

1. In a triangle, the sum of the interior angles is 180°.
2. No triangles can have two obtuse angles.
3. The sine function has a range of $-1 \le \sin\theta \le 1$.
4. If the $\sin\theta$ = positive decimal < 1, the θ can lie in ✓ the first quadrant (acute <) or in the second quadrant (obtuse <).



sin 30 = sin 150!

Example 1: In $\triangle ABC$, $a = 20$, $c = 16$, and $m\angle A = 30°$. How many distinct triangles can be drawn given these measurements?



Use the Law of Sines:

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{20}{\sin 30°} = \frac{16}{\sin C}$$

$$20(\sin C) = 16 \cdot \sin 30°$$

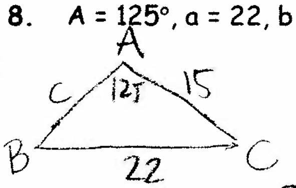
$$\sin C = \frac{16 \cdot (0.5)}{20} = 0.4$$

$C = \sin^{-1}(0.4) = 24°$ (to the *nearest degree*) - in Quadrant I.
Sine is also positive in Quadrant II. If we use the reference angle 24° in Quadrant II, the angle C is 156°.
But, with $m\angle A = 30°$ and $m\angle C = 156°$ the sum of the angles would exceed 180°.
Not possible!!!!

Therefore, $m\angle C = 24°$, $m\angle A = 30°$, and $m\angle B = 126°$ and only **ONE triangle** is possible.

SSA!

8. A = 125°, a = 22, b = 15



$\angle B + \angle C = 55$

$B = 33°57'$

$\frac{\sin 125}{22} = \frac{\sin B}{15} = \frac{\sin 21°03'}{c}$

$\frac{15 \sin 125}{22} = \sin B$

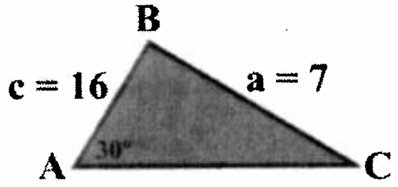
$33.95° = B$

$C = 179°60' - 158°57' = 21°03'$

Since $\angle B + \angle C = 55°$, $\angle B$ could not be obtuse, so there's only 1 Δ!

$c = \frac{22 \sin 21°03'}{\sin 125} \approx 9.65$

Example 2: In $\triangle ABC$, $a = 7$, $c = 16$, and $m\angle A = 30°$. How many distinct triangles can be drawn given these measurements?



Use the Law of Sines:

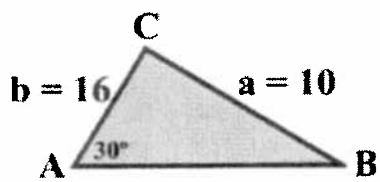
$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{7}{\sin 30°} = \frac{16}{\sin C}$$

$$7(\sin C) = 16 \cdot \sin 30°$$

$$\sin C = \frac{16 \cdot (0.5)}{7} = 1.1428$$

Since $\sin C$ must be ≤ 1, no angle exists for angle C.
NO triangle exists for these measurements.

Example 3: In $\triangle ABC$, $a = 10$, $b = 16$, and $m\angle A = 30°$. How many distinct triangles can be drawn given these measurements?



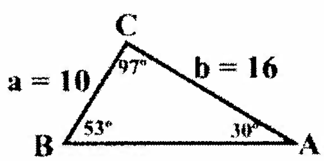
Use the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{10}{\sin 30°} = \frac{16}{\sin B}$$

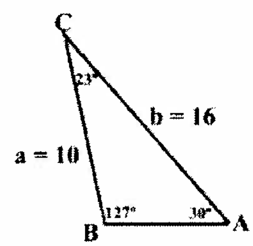
$$10(\sin B) = 16 \cdot \sin 30°$$

$$\sin B = \frac{16 \cdot (0.5)}{10} = 0.8$$

$B = \sin^{-1}(.8) = 53.13010 = 53°$.
Angles could be 30°, 53°, and 97° : sum 180°
The angle from Quadrant II could create angles 30°, 127°, and 23° : sum 180°

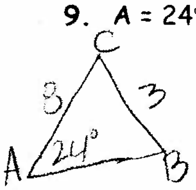


TWO triangles possible.



This example is the **Ambiguous Case.** The information given is the postulate SSA (or ASS, the Donkey Theorem), but the two triangles that were created are clearly not congruent. We have two triangles with two sides and the non-included angle congruent, but the triangles are not congruent to each other.

9. A = 24°, a = 3, b = 8



$\frac{\sin 24}{3} = \frac{\sin B}{8}$

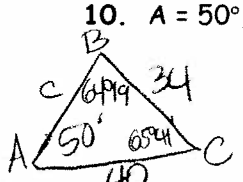
$\frac{8 \sin 24}{3} = \sin B$

$1.08 = \sin B$

sin B must be ≤ 1, so no Δ exists!

10. A = 50°, a = 34, b = 40

C = 179 60 − 114 19 = 65°41



$\frac{\sin 50}{34} = \frac{\sin B}{40} = \frac{\sin 65°41}{c}$

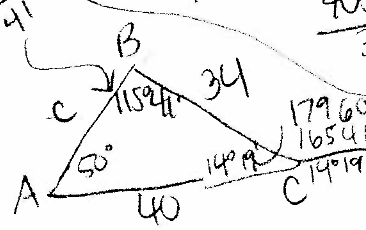
$\frac{40 \sin 50}{34} = \sin B = 64.32°$

64°19'

$c = \frac{34 \sin 65°41'}{\sin 50} \approx 40.45$

or

179 60 − 64 19 = 115 41



179 60 − 165 41 = 14°19

$\frac{\sin 14°19}{c} = \frac{\sin 50}{34}$

$c = \frac{34 \sin 14°19}{\sin 50} \approx 10.95$