14.1 Trigonometric Identities

Trigonometric Identity: An equation involving trigonometric functions that that is true for all values for which every expression in the equation is defined.

KeyConcept Basic Trigonometric Identities

Quotient Identities		
$\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$		$\cot\theta = \frac{\cos\theta}{\sin\theta}$, $\sin\theta \neq 0$
Reciprocal Identities		
$\sin\theta = \frac{1}{\csc\theta}$, $\csc\theta \neq 0$		$\csc\theta = \frac{1}{\sin\theta}$, $\sin\theta \neq 0$
$\cos\theta = \frac{1}{\sec\theta}$, $\sec\theta \neq 0$		$\sec\theta = \frac{1}{\cos\theta}$, $\cos\theta \neq 0$
$\tan\theta = \frac{1}{\cot\theta}$, $\cot\theta \neq 0$		$\cot\theta = \frac{1}{\tan\theta}$, $\tan\theta \neq 0$
Pythagorean Identities		
$\cos^2\theta + \sin^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$	$\cot^2\theta + 1 = \csc^2\theta$
Cofunction Identities		
$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
Negative Angle Identities		
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$

The negative angle identities are sometimes called *odd-even* identities.

Trigonometric Identities can be used to find exact values of trigonometric functions.

Objective 1: The student will be able to find the values of the other five trigonometric functions of ϑ.

Example: *Follow the steps below:* **Given:** $\tan\vartheta = -\frac{2}{3}$ **and** $\frac{3\pi}{2} \le \vartheta \le 2\pi$

1. Write the Pythagorean Identity: $1 + \tan^2\vartheta = \sec^2\vartheta$
2. Substitute the value in for ϑ: $1 + \left(-\frac{2}{3}\right)^2 = \sec^2\vartheta$
3. Simplify: $\frac{13}{9} = \sec^2\vartheta$
4. Take the square root of each side: $\pm\frac{\sqrt{13}}{3} = \sec\vartheta$
5. Because ϑ is in the 4th Quadrant, sec ϑ is positive! $\frac{\sqrt{13}}{3} = \sec\vartheta$
6. Now that you know tan ϑ and sec ϑ, you can find the other four:

☺ Since $\sec\vartheta = \frac{1}{\cos\vartheta}$, then $\cos\vartheta = \frac{3}{\sqrt{13}} \bullet \frac{\sqrt{13}}{\sqrt{13}} =$ $\frac{3\sqrt{13}}{13} = \cos\vartheta$

☺ Since $\cot\vartheta = \frac{1}{\tan\vartheta}$, then $\cot\vartheta =$ $-\frac{3}{2} = \cot\vartheta$

☺ Since $\tan\vartheta = \frac{\sin\vartheta}{\cos\vartheta} \rightarrow \frac{-2}{3} = \frac{\sin\vartheta}{\frac{3}{\sqrt{13}}}$, then $\sin\vartheta = \frac{-2}{3} \bullet \frac{3}{\sqrt{13}} \bullet \frac{\sqrt{13}}{\sqrt{13}} =$ $-\frac{2\sqrt{13}}{13} = \sin\vartheta$

☺ Since $\csc\vartheta = \frac{1}{\sin\vartheta}$, then $\csc\vartheta = \frac{13}{-2\sqrt{13}} \bullet \frac{\sqrt{13}}{\sqrt{13}} =$ $-\frac{\sqrt{13}}{2} = \csc\vartheta$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ *$\sin\theta = \tan\theta \cdot \cos\theta$

1-6: Find the other 5 trigonometric values using the given value.

Q2 **1.** $\sin\vartheta = \frac{1}{2},\ \frac{\pi}{2} < \vartheta < \pi$

$\sin^2\theta + \cos^2\theta = 1$

$\frac{1}{4} + \cos^2\theta = 1$

$\sqrt{\cos^2\theta} = \sqrt{\frac{3}{4}}$

$\cos\theta = \pm\frac{\sqrt{3}}{2}$ (Q2)

a. $\cos\theta = -\frac{\sqrt{3}}{2}$

b. $\tan\theta = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{-1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$

c. $\sec\theta = \frac{-2}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

d. $\csc\theta = 2$

e. $\cot\theta = -\sqrt{3}$

2. $\cos\vartheta = \frac{-2}{3},\ \pi < \vartheta < \frac{3\pi}{2}$ Q3

$\sin^2\theta + \frac{4}{9} = 1$

$\sin^2\theta = \frac{5}{9}$

$\sin\theta = \pm\frac{\sqrt{5}}{3}$

Q3: a. $\sin\theta = -\frac{\sqrt{5}}{3}$

b. $\tan\theta = \frac{-\frac{\sqrt{5}}{3}}{-\frac{2}{3}} = \frac{\sqrt{5}}{2}$

c. $\sec\theta = -\frac{3}{2}$

d. $\csc\theta = -\frac{3}{\sqrt{5}} = -\frac{3\sqrt{5}}{5}$

e. $\cot\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$

Q1 **3.** $\tan\vartheta = \frac{3}{4},\ 0 < \vartheta < \frac{\pi}{2}$

$1 + \tan^2\theta = \sec^2\theta$

$1 + \frac{9}{16} = \sec^2\theta$

$\frac{25}{16} = \sec^2\theta$

$\pm\frac{5}{4} = \sec\theta$

Q1: a. $\sec\theta = \frac{5}{4}$

b. $\cos\theta = \frac{4}{5}$

c. $\cot\theta = \frac{4}{3}$

*d. $\sin\theta = \frac{3}{4} \cdot \frac{4}{5} = \frac{3}{5}$

e. $\csc\theta = \frac{5}{3}$

4. $\sec\vartheta = \frac{-6}{5},\ \frac{\pi}{2} < \vartheta < \pi$ Q2

$1 + \tan^2\theta = \frac{36}{25}$

$\tan^2\theta = \frac{11}{25}$

$\tan\theta = \pm\frac{\sqrt{11}}{5}$

Q2: a. $\tan\theta = -\frac{\sqrt{11}}{5}$

b. $\cot\theta = \frac{-5}{\sqrt{11}} = -\frac{5\sqrt{11}}{11}$

c. $\cos\theta = -\frac{5}{6}$

d.* $\sin\theta = -\frac{\sqrt{11}}{5} \cdot -\frac{5}{6} = \frac{\sqrt{11}}{6}$

e. $\csc\theta = \frac{6}{\sqrt{11}} = \frac{6\sqrt{11}}{11}$

Q4 **5.** $\csc\vartheta = \frac{-5}{3},\ \frac{3\pi}{2} < \vartheta < 2\pi$

a. $\sin\theta = -\frac{3}{5}$

$\frac{9}{25} + \cos^2\theta = 1$

$\cos^2\theta = \frac{16}{25}$

$\cos\theta = \pm\frac{4}{5}$

b. Q4: $\cos\theta = \frac{4}{5}$

c. $\sec\theta = \frac{5}{4}$

d. $\tan\theta = \frac{-\frac{3}{5}}{\frac{4}{5}} = -\frac{3}{4}$

e. $\cot\theta = -\frac{4}{3}$

6. $\cot\vartheta = \frac{1}{2},\ \pi < \vartheta < \frac{3\pi}{2}$ Q3

a. $\tan\theta = 2$

$1 + 4 = \sec^2\theta$

$5 = \sec^2\theta$

$\pm\sqrt{5} = \sec\theta$

Q3: b. $\sec\theta = -\sqrt{5}$

c. $\cos\theta = \frac{-1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$

d.* $\sin\theta = 2 \cdot -\frac{\sqrt{5}}{5} = -\frac{2\sqrt{5}}{5}$

e. $\csc\theta = \frac{-5}{2\sqrt{5}} = -\frac{5\sqrt{5}}{10} = -\frac{\sqrt{5}}{2}$

Objective 2: The student will be able to simplify a trigonometric expression.

Example: *Follow the steps below:* **Given: csc(-x) - csc(-x)cos²x**

1. Factor out csc(-x): $\csc(-x)(1 - \cos^2 x)$
2. Pythagorean Identity: $\csc(-x)(\sin^2 x)$
3. Reciprocal Identity: $\frac{1}{\sin(-x)} \bullet \sin^2 x$
4. Negative Angle Identity: ***(sin(-x) = -sinx)*** $-\frac{1}{\sin x} \bullet \sin^2 x$
5. Simplify: $-\sin x$

7-12: Simplify each trigonometric expression completely.

7. $\frac{\csc^2\vartheta}{1 + \tan^2\vartheta}$

$\frac{\csc^2\theta}{\sec^2\theta}$ $\frac{\frac{1}{\sin^2\theta}}{\frac{1}{\cos^2\theta}}$

$\frac{1}{\sin^2\theta} \cdot \frac{\cos^2\theta}{1} \rightarrow \frac{\cos^2\theta}{\sin^2\theta} = \cot^2\theta$

8. $\frac{\csc\vartheta}{\sec\vartheta}$

$\frac{\frac{1}{\sin\theta}}{\frac{1}{\cos\theta}} = \frac{1}{\sin\theta} \cdot \frac{\cos\theta}{1} = \frac{\cos\theta}{\sin\theta} = \cot\theta$

9. $(1 + \cos\vartheta)(1 - \cos\vartheta)$

$1 - \cos^2\theta$

$= \sin^2\theta$

10. $\frac{\tan^2\vartheta}{\sec\vartheta}$

$\frac{\frac{\sin^2\theta}{\cos^2\theta}}{\frac{1}{\cos\theta}} = \frac{\sin^2\theta}{\cos^2\theta} \cdot \frac{\cos\theta}{1}$

$= \frac{\sin^2\theta}{\cos\theta} = \sin\theta \bullet \tan\theta$

11. $1 - 2\sin^2\vartheta$

$1 - \sin^2\theta - \sin^2\theta$

$\cos^2\theta - \sin^2\theta$

12. $\cos^2\vartheta(1 + \tan^2\vartheta)$

$\cos^2\theta(\sec^2\theta)$

$\cos^2\theta \cdot \frac{1}{\cos^2\theta} = 1$