6-6 The Remainder & Factor Theorems

INTRODUCTION:

1. Given $f(x) = x^3 - 8$. Divide $f(x)$ by $x - 2$

~~Do Long Division:~~

$$x-2 \overline{\big)\, x^3+0x^2+0x-8}$$ quotient (x^2+2x+4)

x^3-2x^2

$2x^2+0x$

$2x^2-4x$

$4x-8$

$4x-8$

Do Synthetic Division:

2	1	0	0	-8
	↓	2	4	8
	1	2	4	0

Find f(2):

2^3-8

$8-8 = 0$

***Since there is no remainder, that means x – 2 is a** factor **of $x^3 - 8$, so $x^3 - 8 = (x - 2)($** x^2+2x+4 **)**

2. Given $f(x) = 2x^4 + 3x^3 + 5x - 4$. Divide $f(x)$ by $x + 3$

Do Synthetic Division:

-3	2	3	0	5	-4
	↓	-6	9	-27	66
	2	-3	9	-22	62

Find f(-3):

$2(-3)^4+3(-3)^3+5(-3)-4$

$2(81)+3(-27)+-15-4$

$162-81-19 = 62$

***Since there is no remainder, that means x + 3 is** NOT **a factor of $2x^4 + 3x^3 + 5x - 4$.**

***When you find a value of a function, if the result is** 0**, then the value is a** zero **of the function.**

REMAINDER THEOREM:

If a polynomial P(x) is divided by x – r, the remainder is a constant P(r), and **P(x) = (x – r) · Q(x) + P(r)**

***In #2 above, $2x^4 + 3x^3 + 5x - 4 = (x + 3)($** $2x^3-3x^2+9x-22$**) +** 62

$x(2x^3-3x^2+9x-22)+3(2x^3-3x^2+9x-22)+62$

$2x^4-3x^3+9x^2-22x+6x^3-9x^2+27x-66+62$

$2x^4+3x^3+5x-4$

SYNTHETIC SUBSTITUTION:

When you apply the Remainder Theorem using synthetic division to evaluate a function.

Example: Find f(-1) and f(2) for the given functions.

3. $f(x) = x^3 + 2x^2 + 5$

f(-1) = $-1+2(1)+5 = 6$

f(2) = $8+8+5 = 21$

4. $f(x) = x^4 - 2x^2 - 8 =$ $(x-2)(x^3+2x^2+2x+4)$

f(-1) = $1-2(1)-8 = -11$

f(2) = $16-2(4)-8$

$16-8-8 = \boxed{0}$

2	1	0	-2	0	-8
	↓	2	4	4	8
	1	2	2	4	0

DEPRESSED POLYNOMIAL: When you divide a polynomial by one of its binomial factors, the quotient (result) has this name because it is one degree less than the original polynomial.

FACTOR THEOREM: When you divide a binomial into a polynomial, it is a factor iff the remainder is zero.

The binomial x – r is a factor of the polynomial P(x) if and only if P(r) = 0

Example: Given a polynomial and one of its factors, find the remaining factors of the polynomial.

5. $x^3 + x^2 - 5x + 3$; $x - 1$

1	1	1	-5	3
	↓	1	2	-3
	1	2	-3	0

$(x-1)(x^2+2x-3)$

$(x-1)(x+3)(x-1)$

6. $2x^3 + x^2 - 5x + 2$; $x + 2$

-2	2	1	-5	2
	↓	-4	6	-2
	2	-3	1	0

$(x+2)(2x^2-3x+1)$

$(x+2)(x-1)(2x-1)$

$\frac{2}{-2}\Big|\frac{-3}{-1}$

$(x-\frac{2}{2})(x-\frac{1}{2})$

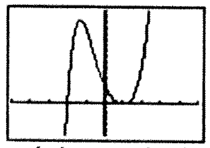
$(x-1)(2x-1)$

6-7 Roots and Zeros

DETERMINING THE NUMBER OF ZEROS OF A FUNCTION:
An n^{th} – degree polynomial function has exactly n zeros. This includes real and imaginary solutions with repeated solutions counted individually.

REAL ZEROS: the points where the graph crosses the x-axis, where y = 0.
If they are not evident, use the 2nd **CALC/ZERO function on the calculator to find them.*
IMAGINARY ZEROS: are determined by finding the difference between the total number of zeros (n) and the number of real zeros.

Example: $f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6$ ⟶



The degree is 5, so there are **5 zeros.**
From the graph we can see that this function has **three real zeros** at –2, 1 (1 is a tangent/double root). Therefore, there are **two imaginary zeros.**
You can find them using synthetic division (the quick way to factor☺).

HOW TO SOLVE A POLYNOMIAL FUNCTION OVER THE SET OF COMPLEX NUMBERS:
1. The quickest way to solve a polynomial function is to factor it. Check first to see if it factors!
2. Then set each factor equal to zero and solve for each x value.
3. State the real and imaginary zeros. Make sure the number of zeros equals the degree of the polynomial function.
4. If the function doesn't factor, type the function into the y_1 screen and find the real zeros.
5. Then use synthetic division and the real zeros to depress the polynomial down to degree 2.
6. Write the function in factored form.
7. Finally, solve the quadratic equation to finish finding your roots.

Example 1: $f(x) = 2x^3 + 11x^2 + 18x + 9$ Given: $f(-3) = 0$
If f(-3) = 0, then -3 is a root. Therefore, (x + 3) is one of the factors of the given polynomial. Use –3 as the divisor and factor the polynomials using synthetic division. This will help you to find the other factors.

$$\begin{array}{r|rrrr} -3 & 2 & 11 & 18 & 9 \\ & \downarrow & -6 & -15 & -9 \\ \hline & 2 & 5 & 3 & 0 \end{array}$$

*Use the same process for synthetic division as with synthetic substitution.

So $(x + 3)(2x^2 + 5x + 3)$ are factors of the given polynomial. Continue factoring the trinomial to find that when completely factored, $2x^3 + 11x^2 + 18x + 9 = (x + 3)(2x + 3)(x + 1)$. So if $(x + 3)(2x + 3)(x + 1) = 0$, then the zeros of the function are: -3, $-\frac{3}{2}$, and -1.

Example 2: Finish the example above: $f(x) = x^5 - 2x^4 + 8x^2 - 13x + 6$): 10
We already know the real zeros are -2, 1, and 1, so depress the polynomial one degree at a time until you get a quadratic function, which will be solved using the quadratic formula.

$$\begin{array}{r|rrrrr|r} -2 & 1 & -2 & 0 & 8 & -13 & 6 \\ & \downarrow & -2 & 8 & -16 & 16 & -6 \\ \hline 1 & 1 & -4 & 8 & -8 & 3 & 0 \\ & \downarrow & 1 & -3 & 5 & -3 & \\ \hline 1 & 1 & -3 & 5 & -3 & 0 & \\ & \downarrow & 1 & -2 & 3 & & \\ \hline & 1 & -2 & 3 & 0 & & \end{array}$$

$(x+2)(x^4-4x^3+8x^2-8x+3)=0$
$(x+2)(x-1)(x^3-3x^2+5x-3)=0$
$(x+2)(x-1)(x-1)(x^2-2x+3)=0$

$x = -2, 1, 1, 1 \pm i\sqrt{2}$

$x = \frac{2 \pm \sqrt{4-4(3)}}{2}$

$\frac{2 \pm \sqrt{-8}}{2}$

$\frac{2 \pm 2i\sqrt{2}}{2}$

Find all zeros of each function:

1. $f(x) = 3x^3 - 11x^2 - 6x + 8$

$$\begin{array}{r|rrrr} -1 & 3 & -11 & -6 & 8 \\ & \downarrow & -3 & 14 & -8 \\ \hline & 3 & -14 & 8 & 0 \end{array}$$

$=(x+1)(3x^2-14x+8)$

$x = \frac{14 \pm \sqrt{196-4(24)}}{6}$

$\frac{14 \pm \sqrt{100}}{6}$

$\frac{14+10}{6}$ $\frac{14-10}{6}$

$\frac{24}{6}, \frac{4}{6}$

$\boxed{x = -1, 4, \frac{2}{3}}$

2. $f(x) = 3x^4 + x^3 - 8x^2 - 2x + 4$

$$\begin{array}{r|rrrrr} -1 & 3 & 1 & -8 & -2 & 4 \\ & \downarrow & -3 & 2 & 6 & -4 \\ \hline & 3 & -2 & -6 & 4 & 0 \end{array}$$

$=(x+1)(3x^3-2x^2-6x+4)$

$=(x+1)(x^2(3x-2)-2(3x-2))$

$=(x+1)(3x-2)(x^2-2)$

$\boxed{x = -1, \frac{2}{3}, \pm\sqrt{2}}$

3. $f(x) = x^4 - 2x^3 + 10x^2 - 18x + 9$

$$\begin{array}{r|rrrrr} 1 & 1 & -2 & 10 & -18 & 9 \\ & \downarrow & 1 & -1 & 9 & -9 \\ \hline & 1 & -1 & 9 & -9 & 0 \end{array}$$

$(x-1)(x^3-x^2+9x-9)$

$(x-1)(x^2(x-1)+9(x-1))$

$=(x-1)(x-1)(x^2+9)$

$\boxed{x = 1, 1, \pm 3i}$

4. $f(x) = 2x^5 - 4x^4 - 2x^3 + 28x^2$

$$\begin{array}{r|rrrr} -2 & 1 & -2 & -1 & 14 \\ & \downarrow & -2 & 8 & -14 \\ \hline & 1 & -4 & 7 & 0 \end{array}$$

$(x+2)(x^4-4x^3+7x^2)$

$(x+2)(x^2)(x^2-4x+7)$

$x = \frac{4 \pm \sqrt{16-4(7)}}{2}$

$\frac{4 \pm \sqrt{-12}}{2}$

$\frac{4 \pm 2i\sqrt{3}}{2}$

$2 \pm i\sqrt{3}$

$\boxed{x = -2, 0, 0, 2 \pm i\sqrt{3}}$

HOW TO WRITE A POLYNOMIAL FUNCTION IN FACTORED FORM THAT HAS THE GIVEN ZEROS:

1. Write the factors out that would create the roots that are given. Keep in mind that all imaginary roots come in pairs. If only one is listed, there is another that is the opposite of the one given.
2. Multiply the factors together two at a time.
3. Write your answer correctly: it is an equation! (y = polynomial or f(x) = polynomial)

Examples:

5. $x = \{-4, 0, 4, 2\}$

$f(x) = (x+4)(x)(x-4)(x-2)$

$x(x^2-16)(x-2)$

$(x^3-16x)(x-2)$

$\boxed{f(x) = x^4 - 2x^3 - 16x^2 + 32x}$

6. $x = \{3i, -\frac{3}{2}\}$

$f(x) = (x-3i)(x+3i)(x+\frac{3}{2})$

$(x^2+9)(2x+3)$

$\boxed{f(x) = 2x^3 + 3x^2 + 18x + 27}$

7. $x = \{8, i, -i\}$

$f(x) = (x-8)(x-i)(x+i)$

$(x-8)(x^2+1)$

$f(x) = x^3 + x - 8x^2 - 8$

$\boxed{f(x) = x^3 - 8x^2 + x - 8}$

$x^4 - 2x^3 + 2x^2$
$-5x^3 + 10x^2 - 10x$
$+6x^2 - 12x + 12$

$x^4 - 7x^3 + 18x^2 - 22x + 12$

8. $x = \{3, 2, 1 + i\}$

$(x-3)(x-2)(x-(1+i))(x-(1-i))$

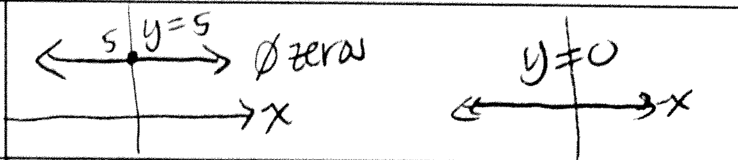
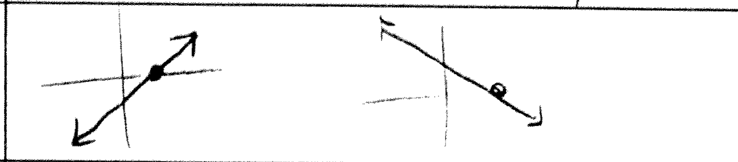
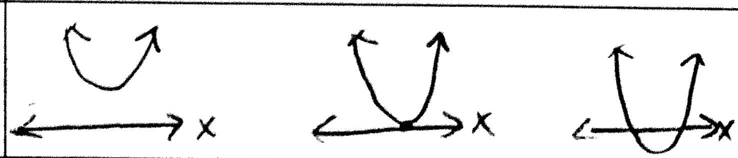
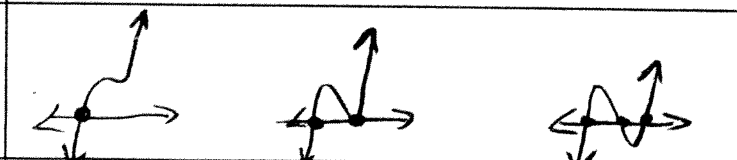
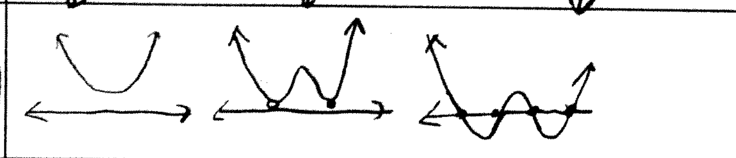
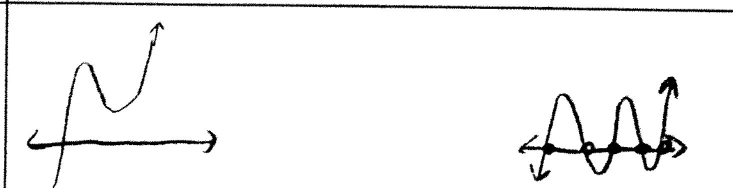
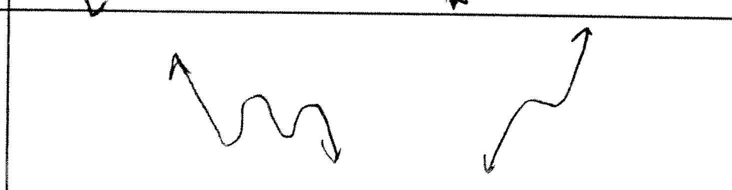
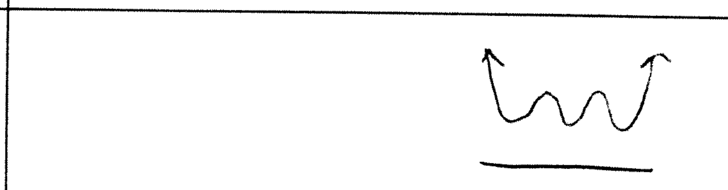
$(x^2-5x+6)(x-1-i)(x-1+i)$

$x^2 - x + ix$
$-x + 1 - i$
$-ix - i^2 + i$

$(x^2-5x+6)(x^2-2x+2)$

$\boxed{f(x) = x^4 - 7x^3 + 18x^2 - 22x + 12}$

SUMMARY OF POLYNOMIAL FUNCTIONS – WHAT WILL THE GRAPH LOOK LIKE?

FUNCTION TYPE DEGREE *Equation*	Even or Odd	POSSIBLE GRAPHS	POSSIBLE # OF REAL ZEROS	POSSIBLE # OF IMAGINARY ZEROS
Constant Degree 0 *f(x) = c*			0 or ∞	
Linear Degree 1 *f(x) = ax + c*	odd		1	0
Quadratic Degree 2 *f(x) = ax²+x+c*	even		0-2	0 or 2
Cubic Degree 3 *f(x) = ax³+x²+x+c*	odd		1-3	0,2
Quartic Degree 4 *f(x) = ax⁴+x³+...+c*	even		0-4	0,2,4
Quintic Degree 5 *f(x) =ax⁵+x⁴+...+c*	odd		1-5	0,2,4
Sextic (Hextic) Degree 6 *f(x) =ax⁶+x⁵+...+c*	even		0-6	0,2,4,6
Septic (Heptic) Degree 7 *f(x) =ax⁷+x⁶+...+c*	odd		1-7	0,2,4,6
Octic Degree 8 *f(x) =ax⁸+x7+...+c*	even		0-8	0,2,4,6,8

all imag. zeros must be in pairs!
even degree can have all imag zeros
odd degree has to have at least 1 real zero